

Numerical Integration

General 1-dimensional formula, $f : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E}(f(x)) \doteq \sum_i^Q w_i f(x_i),$$

where x_i are nodes and w_i are weights.

General 2-dimensional formula (product rule), $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$\mathbb{E}(f(x_1, x_2)) \doteq \sum_i^Q \sum_j^Q w_i w_j f(x_i, x_j). \quad (1)$$

From now on: Assume bivariate **normal** distributed x 's,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathcal{N} \left(\begin{pmatrix} \mu_1 \\ \mu_1 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \right).$$

Some Methods

Monte Carlo: Draw S values of $\{x_1, x_2\}$ from $\mathcal{N}_2(\mu, \Omega)$ and let $w_i = w_j = 1/\sqrt{S}$ to get

$$\mathbb{E}(f(x_1, x_2)) \doteq \frac{1}{S} \sum_s^S f(x_{1,s}, x_{2,s})$$

Quadrature (Product rule): look up $\{\tilde{x}\}$ and $\{w\}$ from a table (more accurate than calculating them!) and transform $\{\tilde{x}\}$ into $\{x_1, x_2\}$ using $\mathcal{N}_2(\mu, \Omega)$ and apply (1)

Sparse Grids: (or “Smolyak’s method”) look up $\{\tilde{x}\}$ and $\{w\}$ from another table (for a given degree of accuracy) and transform $\{\tilde{x}\}$ into $\{x_1, x_2\}$ using $\mathcal{N}_2(\mu, \Omega)$ and apply (1)

Monomial Methods: (or “Stroud’s method”) some different ideas to integrating normal random variables with very few points. See, e.g., Judd (1998). I have found some from http://people.sc.fsu.edu/~jburkardt/m_src/stroud/stroud.html.

Quadrature and Sparse Grids (Heiss and Winschel, 2008)

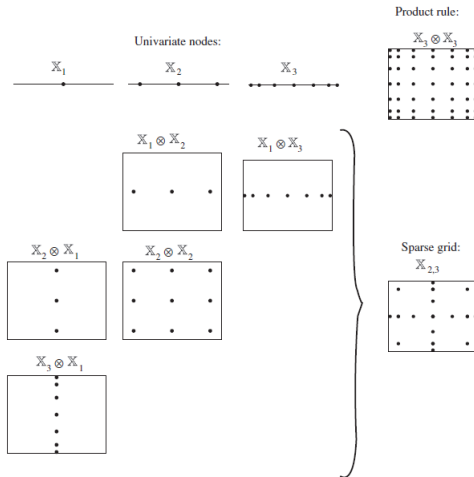


Fig. 1. Construction of the sparse grid in two dimensions.

Constructing nodes/draws

To construct nodes or draws (that all stem from standard normal distribution), we need to add the mean and multiply by the square root of the covariance matrix, $\Omega^{\frac{1}{2}}$.

Unfortunately, the square root of matrix is **not unique!** Some are:

- ▶ **Cholesky decomposition:** $\Omega = LL' \rightarrow \Omega^{\frac{1}{2}} = L$, L is lower cholesky factor,
- ▶ **Eigenvalues:** $\Omega = V\Lambda V \rightarrow \Omega^{\frac{1}{2}} = V\Lambda^{\frac{1}{2}}$, V is a eigenvector and Λ is a diagonal matrix of eigenvalues,
- ▶ **Arbitrary rotation:** $\Omega = V\Lambda V \rightarrow \Omega^{\frac{1}{2}} = VR\Lambda^{\frac{1}{2}}$, V is a eigenvector, R is a “rotation” and Λ is a diagonal matrix of eigenvalues,

Hence, multiple different methods result in surprisingly different “sampling schemes”. Matlab example (MultiDimIntegration.m) inspired by Jäckel (2005) illustrate these methods for all four integration approaches.

References

- Heiss, F. and V. Winschel (2008): “Likelihood approximation by numerical integration on sparse grids,” *Journal of Econometrics*, 144(1), 62 – 80.
- Jäckel, P. (2005): “A note on multivariate Gauss-Hermite quadrature,” Working paper.
- Judd, K. (1998): *Numerical Methods in Economics*. MIT Press, Cambridge.